*Theorems in the Calculus of Enlargement.**

BY EMORY MCCLINTOCK.

In my *Essay on the Calculus of Enlargement* (*Am. Journal of Mathematics*, II, 101–161), that calculus was described, from one point of view, as an extension of the Calculus of Finite Differences, comprising, as its most important branch, the Differential Calculus. I argued that the operation of Enlargement, indicated by

$$\mathrm{E}^h\phi x = \phi(x+h),$$

is simpler than that of Differentiation,

$$\mathrm{D}\phi x = \frac{\phi(x+h)-\phi x}{h}_{[h=0]};$$

that the two operations, E and D, are functions of each other; that whichever is defined last must be expressed in terms of the other; that D should therefore be defined in terms of E, namely, $\mathrm{D} = \log \mathrm{E}$; and that the theory of the functions of E, or Calculus of Enlargement, is a formal algebra, of which the theory of differentiation is that part which corresponds to the theory of logarithms in ordinary algebra. Spontaneous expressions of approval of these suggestions were sent to me by eminent mathematicians of different countries, and I cannot doubt that the ideas in question, being founded in reason, will in time find general acceptance.

In that Essay I gave incidentally (p. 146) several substitutes for Taylor's theorem, by which the coefficients were exhibited in the language of finite differences, or, as I prefer to say, of the calculus of enlargement, without reference to the theory of differentiation. My present purpose is to present another similar series, corresponding to Taylor's theorem, with a more direct proof, and with suitable illustrations, and afterwards to exhibit series corresponding to Lagrange's and Laplace's theorems. In doing this, several symbolic

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expansions of wide scope will be developed. Numbers preceded by the letter *E* will, wherever they occur, be understood to refer to the numbered equations of the Essay in Vol. II.

Let $\qquad \mathrm{B} = \mathrm{E} - z,$

so that $\qquad \mathrm{B}\phi x = \phi(x+1) - z\phi x,$

and let $\qquad x^{m)} = x(x-1)(x-2)\ldots(x-m+1),$

as in *E* 271. Then*

$$\begin{aligned} \mathrm{B}z^{x-m}x^{m)} &= z^{x+1-m}(x+1)^{m)} - z^{x+1-m}x^{m)} \\ &= z^{x+1-m}(x+1-x+m-1)\,x^{m-1)} \\ &= mz^{x-m+1}x^{m-1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{B}^2z^{x-m}x^{m)} &= m(m-1)\,z^{x-m+2}x^{m-2)}, \\ \mathrm{B}^nz^{x-m}x^{m)} &= m^{n)}z^{x-m+n}x^{m-n)} \\ \mathrm{B}^mz^{x-m}x^{m)} &= m\,!\;z^x. \end{aligned}$$

It will be observed that $\mathrm{B}^nz^{x-m}x^{m)} = 0$ when $n > m$. Let $\phi\mathrm{E}$ be any function of E which can be expressed in positive integral powers of B, say,

$$\phi\mathrm{E} = a_0 + a_1\mathrm{B} + a_2\mathrm{B}^2/2\,! + \ldots. \tag{1}$$

Then $\qquad \phi\mathrm{E}z^{x-m}x^{m)} = a_0z^{x-m}x^{m)} + ma_1z^{x-m+1}x^{m-1)} + \ldots.$

Let $x = 0$; then, since $0^{m)} = 0$ for all values of m greater than 0,

$$\phi\mathrm{E}_0z^{0-m}0^{m)} = m^{m)}a_mz^{0-m+m}0^{m-m}/m\,! = a_m.$$

Hence, by substitution in (1),

$$\phi\mathrm{E} = \phi\mathrm{E}_0z^0 + \phi\mathrm{E}_0z^{0-1}0\mathrm{B} + \phi\mathrm{E}_0z^{0-2}0^{2)}\mathrm{B}^2/2\,! + \ldots. \tag{2}$$

This theorem, doubtless new, may be illustrated in various ways. If, for example, it be applied to the problem of interpolation, we may take $z = 1$, $\phi\mathrm{E} = \mathrm{E}^n$, and we have, operating on fx, the well-known formula in finite differences, wherein $\Delta = \mathrm{E} - 1$,

$$f(x+n) = fx + n\Delta fx + n^{2)}\Delta^2fx/2\,! + \ldots.$$

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* It will be understood that operations, functions of E, are indicated by symbols which in each case apply to all that follows in the same expression; while functional symbols, such as ϕ or f, apply only to the letter or bracket which they precede. For example, $\phi\mathrm{E}_xx^mf\mathrm{E}_x\psi x$ means that ψx is operated upon by $f\mathrm{E}_x$, the result multiplied by x^m, and the product operated upon by $\phi\mathrm{E}_x$. Continuity and intelligibility of results (equivalent in the case of series to convergence) are always presupposed.

Let $y = z + k$. We know, by E 112, that $\phi \mathrm{E}_0 y^0 = \phi y$, so that $\mathrm{B}_0^n y^0 = (\mathrm{E}_0 - z)^n y^0 = (y - z)^n = k^n$. If now we operate with both sides of (2) upon y^0 with respect to 0, we have, writing $z + k$ for y,

$$\phi(z + k) = \phi z + \phi \mathrm{E}_0 z^{0-1} 0 . k + \phi \mathrm{E}_0 z^{0-2} 0^{2)} . k^2 / 2\,! + \ldots . \tag{3}$$

Taylor's theorem declares the value of a_m in $\phi(z + k) = a_0 + a_1 k + a^2 k^2 / 2\,! + \ldots .$ to be $\mathrm{D}_z^m \phi z$. This series (3) expresses the value of a_m by the use of the simpler symbol E, without reference to the operation of differentiation.*

In the use of (3) for the expansion of $\phi(z + k)$, the interpretation of the coefficients depends of course upon the form of the function denoted by ϕ. If, for example, $\phi z = z^n$,

$$\mathrm{E}^n z^{0-m} 0^{m)} = z^{n-m} n^{m)},$$

and we obtain the binomial theorem,

$$(z + k)^n = z^n + n z^{n-1} k + \ldots .$$

As another illustration of (3), let $\phi = \log$, and we have for the coefficient of $k^m / m\,!$

$$\begin{aligned} \log \mathrm{E}_0 z^{0-m} 0^{m)} &= 0^{m)} \log \mathrm{E}_0 z^{0-m} + z^{0-} \quad \log \mathrm{E}_0 0^{m)}, \text{ by } E \text{ 135}, \\ &= \log (1 + \Delta_0)\, 0^{m)} z^{-m} \\ &= [\Delta_0 - \tfrac{1}{2} \Delta_0^2 + \tfrac{1}{3} \Delta_0^3 - \ldots .]\, 0^{m)} z^{-m} \\ &= (-1)^{m-1} (m - 1)\,!\, z^{-m}, \end{aligned}$$

since, by E 279, $\Delta^n 0^{m)} = 0$ except when $n = m$, in which case $\Delta^m 0^{m)} = m\,!$ Then (3) becomes

$$\log (z + k) = \log z + z^{-1} k - \tfrac{1}{2} z^{-2} k^2 + \ldots .$$

To interpret a trigonometrical expression, say $\sin \mathrm{E}_0 z^{0-m} 0^{m)}$, we use the analytical definition, say $\sin \mathrm{E} = \mathrm{E} - \mathrm{E}^3 / 3\,! + \ldots .$, and the result will vary according to the value of q in $m = 4p + q$. If, for instance, $m = 1$,

$$\begin{aligned} \sin \mathrm{E}_0 z^{0-1} 0 &= (\mathrm{E}_0 - \mathrm{E}_0^3 / 3\,! + \ldots .)\, z^{0-1} 0 \\ &= 1 - z^2 / 2\,! \quad + \ldots . = \cos z, \end{aligned}$$

and the same if $m = 5$:

$$\begin{aligned} \sin \mathrm{E}_0 z^{0-5} 0^{5)} &= (\mathrm{E}_0 - \mathrm{E}_0^3 / 3\,! + \ldots .)\, z^{0-5} 0^5 \\ &= 5^{5)} / 5\,! - z^2 7^{5)} / 7\,! + \ldots . \\ &= 1 - z^2 / 2\,! + \ldots . = \cos z. \end{aligned}$$

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*In the earlier series, E 318, a_m is represented in the form $z^{-m} \phi (z \mathrm{E}_0)\, 0^m$. The form in (3) is $\phi \mathrm{E}_0 z^{0-m} 0^{m)}$, so that $\phi (z \mathrm{E}_0)\, 0^{m)}$ must be equivalent to $\phi \mathrm{E}_0 z\, 0^{m)}$. It is in fact easy to prove that $\phi (z \mathrm{E}_0)\, \psi 0 = \phi \mathrm{E}_0 z^0 \psi 0$. See E 98.

Again, if $m = 3$, $m = 7$, ..., we have the same result with the opposite sign, viz. $-\cos z$; if $m = 2$, $m = 6$, ..., $-\sin z$; and if $m = 4$, $m = 8$, ..., $+\sin z$. If, therefore, we write sin for ϕ in (3) we obtain

$$\sin(z + k) = \sin z + \cos z . k - \sin z . k^2/2! + \ldots .$$

In more complicated cases the coefficients are to be interpreted by observing general rules (equivalent to the usual rules for differentiation) which may be derived and proved, without reference to limits or differentials, by analytical methods alone, analogous to those laid down by Lagrange in his *Calcul des Fonctions.*

I would not be understood as suggesting this formula and other like formulæ mentioned in the previous paper referred to as improvements upon Taylor's theorem. My object is, however, something more than the mere exhibition of interesting novelties. This series (3) shows at a glance, what indeed has otherwise been abundantly proved, that there exists no barrier, no definite boundary, between the branches known as the Calculus of Finite Differences and the Differential Calculus. The Calculus of Enlargement, or algebra of the functions of E, comprises both those branches; the differential calculus, which relates to $\mathrm{D} = \log \mathrm{E}$, being that part of the symbolic algebra of the functions of E which corresponds to the theory of logarithms in ordinary algebra. Nor would I have thought it useful to present this new series (3) at this time at all, considering the other similar series given in the earlier paper, were it not that it happens to be naturally introductory to the presentation of wider and more important theorems.

The symbolic series (2) is but a special case of this:

$$\phi \mathrm{E} = \phi z + \phi \mathrm{E}_0 z^{0-h} 0 . \mathrm{A} + \phi \mathrm{E}_0 z^{0-2h} 0^{2]} . \mathrm{A}^2/2! + \ldots . \tag{4}$$

Here, as before, $\phi z = \phi \mathrm{E}_0 z^0$; and A represents $(\mathrm{E}^h - z^h)/h$, so that B is what A becomes when $h = 1$; also,

$$x^{m]} = x(x-h)(x-2h) \ldots (x - mh + h),$$

so that $x^{m)}$ is what $x^{m]}$ becomes when $h = 1$; so that, in short, (3) is what (4) becomes when $h = 1$. It is needless to recount the steps, exactly similar to those taken to prove (2), by which we may derive (4); we may note, however, that

$$\mathrm{A}_x z^{x-hm} x^{m]} = m z^{x-h(m-1)} x^{m-1]}.$$

Apart from the case (2) already considered, the most notable special case of (4)

is that wherein $h=0$. Let $h=0$ and $z=e^u$; then since $(x^h-1)/h=\log x$ when $h=0$, $\text{A}=\log\text{E}-\log z=\text{D}-u$, and (4) becomes

$$\phi\text{E}=\phi(e^u)+\phi\text{E}_0e^{u0}0\,(\text{D}-u)+\phi\text{E}_0e^{u0}0^2\,(\text{D}-u)^2/2\,!+\ldots. \tag{5}$$

If $\phi\text{E}=\log\text{E}=\text{D}$, this yields merely the identity $\text{D}=u+\text{D}-u$. If $\phi\text{E}=\text{E}^n$, and if we operate on fx and divide both sides by e^{un}, we derive a curious generalization of Taylor's theorem:

$$e^{-un}f(x+n)=fx+n\,(\text{D}-u)fx+n^2\,(\text{D}-u)^2fx/2\,!+\ldots. \tag{6}$$

When $u=0$, this becomes Taylor's theorem. If in (6), for example, $fx=e^{2ux}$, we have

$$e^{-un}e^{2u\,(x+n)}=e^{2ux}+nue^{2ux}+n^2u^2e^{2ux}/2\,!+\ldots.$$

If $y=e^{u+k}$, and if we operate with (5) upon y^0, remembering that $\phi\text{E}_0y^0=\phi y$, we have this interesting result,

$$\phi\,(e^{u+k})=\phi\,(e^u)+\phi\text{E}_0e^{u0}0\,.k+\phi\text{E}_0e^{u0}0^2.k^2/2\,!+\ldots. \tag{7}$$

or, substituting $\phi\log$ for ϕ,

$$\phi\,(u+k)=\phi u+\phi\text{D}_0e^{u0}0\,.k+\phi\text{D}_0e^{u0}0^2.k^2/2\,!+\ldots. \tag{8}$$

We have here still another substitute for Taylor's theorem, wherein, as will be observed, $\text{D}_u^m\phi u=\phi\text{D}_0e^{u0}0^m$, a relation otherwise derivable at once from E 169, where $\phi\text{D}_x\psi_x=\psi\text{D}_0e^{x0}\phi 0$. If in (7), as a special case, we put $u=0$, we have Herschel's theorem,

$$\phi\,(e^k)=\phi 1+\phi\text{E}_0 0\,.k+\phi\text{E}_0 0^2.k^2/2\,!+\ldots.$$

A more useful form of (7), for some purposes, may be

$$\phi\,(ze^k)=\phi z+\phi\text{E}_0z^0 0\,.k+\phi\text{E}_0z^0 0^2.k^2/2\,!+\ldots. \tag{9}$$

If for any reason we desire to ignore the operation of differentiation, we can determine the successive coefficients of (7) by writing $\phi_m u$ for $\phi\text{E}_0e^{u0}0^m$, and employing the relation

$$\phi_m\text{E}_0u^{0-1}0=\phi\text{E}_0e^{u0}0^{m+1}=\phi_{m+1}u.$$

For, writing v for 0 in $\phi_m u$, to avoid using two kinds of zeros, we have $\phi_m u=\phi\text{E}_ve^{uv}v^m$, and

$$\begin{aligned}\phi_m\text{E}_0u^{0-1}0&=\phi\text{E}_ve^{v\text{E}}v^mu^{0-1}0\\&=\phi\text{E}_vv^m\,(1+v\text{E}_0+\ldots\;)u^{0-1}0\\&=\phi\text{E}_vv^m\,(v+v^2u+v^3u^2/2\,!+\ldots.)\\&=\phi\text{E}_vv^{m+1}e^{uv}\\&=\phi\text{E}_0e^{u0}0^{m+1}=\phi_{m+1}u.\end{aligned}$$

That is to say, having found the value of the m^{th} coefficient as a function of u, $\phi_m u$, we take the same function of E as an operator upon $u^{0-1}0$ to obtain the next coefficient.

The general principle which I have followed, in the earlier paper and in this, in the development of theorems of expansion, consists in assuming that expansion is practicable, and therefore that the form of the coefficients is all that is to be sought; and in finding a series of functions, $f_0x, f_1x, \ldots.$, and an operator P, such that $\mathrm{P}f_m x = mf_{m-1}x$, $\mathrm{P}f_0 x = 0$; whence, as in the proof of (2), it follows that, for any function ϕ which can be expressed in positive integral powers,

$$\phi\mathrm{P} = \phi\mathrm{P}_0 f_0 0 + \phi\mathrm{P}_0 f_1 0 \,.\, \mathrm{P} + \phi\mathrm{P}_0 f_2 0 \,.\, \mathrm{P}^2/2! + \ldots. \tag{10}$$

For the simplest series of functions, x^0, x^1, x^2, ..., the operator is D. Another series of functions is x^0, x, $x(x-h)$, $x(x-h)(x-2h)$, ..., commonly called factorials, and represented here by $x^{0]}$, $x^{1]}$, $x^{2]}$, ... The operator corresponding is $\mathrm{E}^h - 1$, and the resulting series is well known. In the earlier paper I extended the then existing theory by devising the more general form of factorial shown in *E* 265 and *E* 267,

$$x^{[m]} = x(x + amh - h)(x + amh - 2h) \ldots . (x + amh - mh + h),$$

with the corresponding operator $(\mathrm{E}^{-ha+h} - \mathrm{E}^{-ha})/h$. The result was a very general operative series which included as special cases those just mentioned (in which respectively $h = 0$ and $a = 0$) and others more novel, such for example as those in which the operators are $1 - \mathrm{E}^{-h}$, $\mathrm{E}^{\frac{1}{2}h} - \mathrm{E}^{-\frac{1}{2}h}$, DE^c. In the present paper the theorems (2) and (4) correspond to the operators $\mathrm{E} - z$ and $(\mathrm{E}^h - z^h)/h$ respectively, the former a special case of the latter. We have now to consider a still wider generalization involving the same principle of procedure.

Let $\mathrm{A} = (\mathrm{E}^h - z^h)/h$, as before, and let $x^{|m|} = x(\phi\mathrm{E}_x)^m x^{m-1|}$, where $x^{m-1|} = z^{x-mh}(x-h)(x-2h) \ldots . (x - mh + h)$. Also, $x^{|0|} = z^x$, $\mathrm{A}x^{|0|} = 0$. It is needed to prove that $\mathrm{A}x^{|m|} = m\phi\mathrm{E}_x x^{|m-1|}$.

We may in the first place prove that $\mathrm{A}x^{|m|} = m\phi\mathrm{E}_x x^{|m-1|}$ by analysis of the expression $(\phi\mathrm{E})^m$ contained in $x^{|m|}$, on the assumption that $\phi\mathrm{E}$ can be expressed in terms of A. Let $a_\alpha \mathrm{A}^\alpha$ and $a_\beta \mathrm{A}^\beta$ be any terms of $\phi\mathrm{E}$; then $a_\alpha \mathrm{A}^\alpha a_\beta \mathrm{A}^\beta$ will be a component of $(\phi\mathrm{E})^2$. When $\phi\mathrm{E}$ is used a third time as multiplier, let $a_\gamma \mathrm{A}^\gamma$ be any term of it; then $a_\alpha \mathrm{A}^\alpha a_\beta \mathrm{A}^\beta a_\gamma \mathrm{A}^\gamma$ will be a component of $(\phi\mathrm{E})^3$. In general, similarly, $a_\alpha a_\beta \ldots . a_\mu \mathrm{A}^{\alpha+\beta+\cdots+\mu}$ will be a component of $(\phi\mathrm{E})^m$, $a_\mu \mathrm{A}^\mu$ being

any term of ϕE when used as a multiplier for the m^{th} time. Let us denote this component by $a_s\text{A}^s$. The corresponding component of $x^{|m|} = x(\phi\text{E})^m x^{m-1|}$ is therefore $a_s x \text{A}^s x^{m-1|}$. Let us denote this by comp $x^{|m|}$. Then

$$\begin{aligned}\text{A}\ \text{comp}\ x^{|m|} &= \text{A}a_s x\,(m-1)^{s)}\,x^{m-1-s|}\\ &= a_s\,(m-1)^{s)}\,(m-s)\,x\,.\,x^{m-s-2|\,*}\\ &= w\,(m-1)(m-s),\end{aligned} \tag{11}$$

where $w = a_s\,(m-2)^{s-1)}x\,.\,x^{m-s-2|}$. Similarly, comp $(\phi\text{E})^{m-1} = a_s a_\mu^{-1}\text{A}^{s-\mu}$, and

$$\begin{aligned}\text{comp}\ m\phi\text{E}x^{|m-1|} &= \text{comp}\ m\phi\text{E}x\,(\phi\text{E})^{m-1}x^{m-2|}\\ &= ma_\mu\text{A}^\mu x a_s a_\mu^{-1}\text{A}^{s-\mu}x^{m-2|}\\ &= ma_s\text{A}^\mu x\,(m-2)^{s-\mu)}x^{m-s+\mu-2|\,*}\\ &= ma_s\,(m-2)^{s-\mu},(m-s+\mu-1)^{\mu)}x\,.\,x^{m-s+2|}\\ &= wm\,(m-s+\mu-1).\end{aligned} \tag{12}$$

The difference between (11) and (12) is $w(s-m\mu)$. This shows that no component of $\text{A}x^{|m|}$ is necessarily equal to the corresponding component of $m\phi\text{E}x^{|m-1|}$, except in the case where $\alpha = \beta = \ldots . = \mu$, when $s = m\mu$, and the difference vanishes. In all other cases, however, we may so group the components as to find the sum of any group in $\text{A}x^{|m|}$ equal to that of the corresponding group in $m\phi\text{E}x^{|m-1|}$. As the first of the group, take the component already considered. As the next, take that component which is formed as first described, but with a cyclic interchange of factors, namely, with $a_\beta\text{A}^\beta$ in lieu of $a_\alpha\text{A}^\alpha$, $a_\gamma\text{A}^\gamma$ in lieu of $a_\beta\text{A}^\beta$, etc., ending with $a_\alpha\text{A}^\alpha$ in lieu of $a_\mu\text{A}^\mu$. The difference between the components in this case will be $w(s-m\alpha)$. In the next case, proceeding as before, the difference will be $w(s-m\beta)$, and so on, until when the group is completed the sum of the differences becomes $w(sm-ms)=0$. Since the sum of each group in $\text{A}x^{|m|}$ is equal to the sum of the corresponding group in $m\phi\text{E}x^{|m-1|}$, it follows that these two expressions are equivalent.

Another proof that $\text{A}x^{|m|} = m\phi\text{E}_x x^{|m-1|}$ depends on the assumption that ϕE can be expressed in powers of E, not necessarily positive or integral. We remark first that

$$(\phi\text{E})^2\,xfx = 2\phi\text{E}x\phi\text{E}fx - x\,(\phi\text{E})^2 fx. \tag{13}$$

For, the terms of ϕE being of the form $a\text{E}^\alpha$, $b\text{E}^\beta$, ... , each term of $(\phi\text{E})^2$ will be of the form $a^2\text{E}^{2\alpha}$ or $2ab\text{E}^{\alpha+\beta}$. As regards each term of the form $a^2\text{E}^{2\alpha}$, we have the identity

$$a^2\text{E}^{2\alpha}xfx = 2a\text{E}^\alpha xa\text{E}^\alpha fx - xa^2\text{E}^{2\alpha}fx,$$

* As will be seen by performing the operation denoted by A.

because the performance of the operations indicated produces the identity

$$a^2(x+2\alpha)f(x+2\alpha)=2a^2(x+\alpha)f(x+2\alpha)-a^2xf(x+2\alpha).$$

Similarly, as regards each term of the second form,

$$2ab\mathrm{E}^{\alpha+\beta}xfx=2a\mathrm{E}^{\alpha}xb\mathrm{E}^{\beta}fx+2b\mathrm{E}^{\beta}xa\mathrm{E}^{\alpha}fx-2xab\mathrm{E}^{\alpha+\beta}fx.$$

By summing the terms of both forms we obtain (13). Let $fx=(\phi\mathrm{E})^{m-1}\psi x$; then (13) becomes

$$(\phi\mathrm{E})^2x(\phi\mathrm{E})^{m-1}\psi x=2\phi\mathrm{E}x(\phi\mathrm{E})^m\psi x-x(\phi\mathrm{E})^{m+1}\psi x. \tag{14}$$

If the equation

$$(\phi\mathrm{E})^m x\psi x=m\phi\mathrm{E}x(\phi\mathrm{E})^{m-1}\psi x-(m-1)x(\phi\mathrm{E})^m\psi x \tag{15}$$

be true for any value of m, it is true for the value next higher, and so on for all higher values; for operating on (15) by $\phi\mathrm{E}$, and substituting for the first term of the second member its value from (14), we derive at once

$$(\phi\mathrm{E})^{m+1}x\psi x=(m+1)\phi\mathrm{E}x(\phi\mathrm{E})^m\psi x-mx(\phi\mathrm{E})^{m+1}\psi x.$$

But (15) is true, by (13), when $m=2$, and it is therefore true for all higher values. To go back now to $x^{|m|}$, its definition is,

$$x^{|m|}=x(\phi\mathrm{E}_x)^m z^{x-mh}(x-h)(x-2h)\dots(x-mh+h).$$

If $\psi x=z^{x-mh}(x-h)\dots(x-mh+2h)$, we have $x^{|m|}=x(\phi\mathrm{E})^m(x-mh+h)\psi x$. If we operate upon this with $\mathrm{A}=(\mathrm{E}^h-z^h)/h$, we have

$$\begin{aligned}\mathrm{A}x^{|m|}&=[(x+h)(\phi\mathrm{E})^m x\psi x-x(\phi\mathrm{E})^m(x-mh+h)\psi x]/h\\&=(\phi\mathrm{E})^m x\psi x+(m-1)x(\phi\mathrm{E})^m\psi x\\&=m\phi\mathrm{E}x(\phi\mathrm{E})^{m-1}\psi x,\text{ by (15)},\\&=m\phi\mathrm{E}x^{|m-1|}\end{aligned} \tag{16}$$

Let $\mathrm{H}=\mathrm{A}(\phi\mathrm{E})^{-1}=(\phi\mathrm{E})^{-1}\mathrm{A}$, since all functions of E are commutative; then, performing the operation $(\phi\mathrm{E})^{-1}$ upon both sides of (16), we find that

$$\begin{aligned}\mathrm{H}x^{|m|}&=mx^{|m-1|},\\\mathrm{H}^n x^{|m|}&=m^{n)}x^{|m-n|},\\\mathrm{H}^m x^{|m|}&=m^{m)}x^{|0|}=m^{m)}z^x,\\\mathrm{H}^{m+1}x^{|m|}&=0.\end{aligned}$$

Let $f\mathrm{E}$ be any function of E which can be expressed in positive integral powers of H, say

$$f\mathrm{E}=a_0+a_1\mathrm{H}+a_2\mathrm{H}^2/2!+\dots. \tag{17}$$

Then

$$f\mathrm{E}x^{|m|}=a_0x^{|m|}+a_1mx^{|m-1|}+a_2m^{2)}x^{|m-2|}/2!+\dots,$$

whence, if $x=0$, $\qquad f\mathrm{E}0^{|m|}=a_m$.

Substituting this for all values of m in (17), and observing that $f\mathrm{E}0^{|0|}=f\mathrm{E}z^0=fz$, we obtain finally this general symbolic theorem:

$$f\mathrm{E}=fz+f\mathrm{E}_0 0^{|1|}\mathrm{H}+f\mathrm{E}_0 0^{|2|}\mathrm{H}^2/2!+\ldots, \tag{18}$$

where $\mathrm{H}=(\mathrm{E}^h-z^h)/h\phi\mathrm{E}$, and $x^{|m|}=x\,(\phi\mathrm{E}_x)^m z^{x-mh}\,(x-h)(x-2h)\ldots(x-mh+h)$. This comprehensive expression includes as special cases all the series already mentioned in this and the earlier paper as well as those yet to be given. If $\phi\mathrm{E}=\mathrm{E}^{ah}$ and if $z=1$, we have from (18) the "factorial theorem" (*E* 291) which was presented and extensively discussed in the former paper, where many deductions were made from it. A great number of special cases and applications might be deduced from the wider formula (18), but I shall confine attention at present to a few of the more notable cases.

Let $x=(y^h-z^h)/h\phi y$, so that x is the same function of y as H is of E; then, operating with (18) upon y^0 with respect to 0, and remembering that $f\mathrm{E}_0 y^0=fy$, we have the following general series for the expansion of a function of y in terms of x, when $y^h=z^h+xh\phi y$:

$$fy=fz+f\mathrm{E}_0 0^{|1|}.x+f\mathrm{E}_0 0^{|2|}.x^2/2!+f\mathrm{E}_0 0^{|3|}.x^3/3!+\ldots \tag{19}$$

This looks like Lagrange's and Laplace's series, yet it is new, simple and useful, in and of itself. I find no indication that any one hitherto has attempted to expand in terms of x, by either of the series named, a function of y, when $y^h=z^h+xh\phi y$. If we put $y^h=u$, and expand $f(u^{\frac{1}{h}})$ by either series, we find that

$$fy=fz+z^{1-h}\phi zf'z.x+z^{1-h}\frac{d}{dz}\left[z^{1-h}(\phi x)^2f'z\right].x^2/2!$$
$$+\left(z^{1-h}\frac{d}{dz}\right)^2\left[z^{1-h}(\phi z)^3f'z\right].x^3/3!+\ldots \tag{20}$$

This series, also new, is identical with (19) except in form, and may be preferred by those who are not acquainted with, or who find a difficulty in the use of, the notation of the calculus of finite differences. If $fy=y$, we have as special cases of (19) and (20) the following new and highly important developments, identical except in form:

$$y=z+z^{1-h}\phi z.x+\mathrm{E}_0\,(\phi\mathrm{E}_0)^2z^{0-2h}\,(0-h).x^2/2!+\ldots, \tag{21}$$

$$y=z+z^{1-h}\phi z.x+z^{1-h}\frac{d}{dz}\left[z^{1-h}(\phi z)^2\right].x^2/2!+\ldots \tag{22}$$

The manner of employing this biform expression in determining approximately the roots of any equation, $y^h = z^h + hx\phi y$, is shown in the succeeding paper, "On a Method for Calculating Simultaneously all the Roots of an Equation."

If, in (20), $h = 1$, we have Lagrange's well-known theorem, where $y = z + x\phi y$,

$$fy = fz + \phi z f'z . x + \frac{d}{dz}\left[(\phi z)^2 f'z\right].x^2/2! \ldots , \tag{23}$$

corresponding to which we have, from (19),

$$fy = fz + f\mathrm{E}_0 0 \phi \mathrm{E} z^{0-1}.x + f\mathrm{E}_0 0\,(\phi \mathrm{E}_0)^2 z^{0-2}(0-1).x^2/2! + \ldots , \tag{24}$$

in which the coefficients have the same value as in (23), but are expressed without reference to the operation of differentiation. If $\phi\mathrm{E} = \mathrm{E}^0 = 1$, $y = z + x$, and (24) becomes reduced, as a special case, to (3), in which Taylor's theorem is replaced by one in which the coefficients are expressed without reference to differentiation. To obtain a corresponding expression in lieu of Laplace's theorem, which expands fu in terms of x, where $u = \psi(z + x\phi u)$, let $u = \psi y$, and from (24) we have

$$fu = f\psi z + f\psi \mathrm{E}_0 0 \phi\psi \mathrm{E} z^{0-1}.x + f\psi\mathrm{E}_0 0\,(\phi\psi\mathrm{E}_0)^2 z^{0-2}(0-1).x^2/2! \ldots , \tag{25}$$

Laplace's theorem being

$$fu = f\psi z + \phi\psi z \frac{d}{dz} f\psi z . x + \frac{d}{dz}\left[(\phi\psi z)^2 \frac{d}{dz} f\psi z\right].x^2/2! + \ldots .$$

Another new series, akin to (24), may be derived from (19) by putting $h = 0$, in which case $x\phi y = \frac{y^h - z^h}{h}{}_{[h=0]} = \log y - \log z$. That is to say, assuming $z = e^u$, the relation involved is $\log y = u + x\phi y$; and the series is

$$fy = f(e^u) + f\mathrm{E}_0 0 \phi \mathrm{E}_0 e^{u0}.x + f\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^2 e^{u0} 0 . x^2/2! + \ldots , \tag{26}$$

the general term being $f\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^m e^{u0} 0^{m-1}.x^m/m!$. If, as a very simple special case, we put $u = 0$ and $\phi y = 1$, we have Herschel's theorem. If $\log y = v$, and if $\phi y = \psi \log y = \psi v$, we derive from (26), writing $f\log$ for f, still another important result,

$$fv = fu + f\mathrm{D}_0 0 \psi \mathrm{D}_0 e^{u0}.x + f\mathrm{D}_0 0\,(\psi\mathrm{D}_0)^2 e^{u0} 0 . x^2/2! + \ldots . \tag{27}$$

Here $v = u + x\psi v$, and $\mathrm{D}_0 = \log \mathrm{E}_0 = \frac{d}{dx}{}_{[x=0]}$. Comparing this with Lagrange's

theorem (23) and the corresponding series (24), we see that the coefficients of all three must be identical in value. We have here, in fact, another form for the expansion effected by Lagrange's theorem, a form which will be found entirely analogous in principle to the "secondary form of Maclaurin's theorem" given by Boole in the second chapter of his *Finite Differences*:

$$fv = f0 + f\mathrm{D}_0 0 \,.\, v + f\mathrm{D}0^2 \,.\, v^2/2! + \ldots .$$

To reduce (27) to this form we have merely to put $u = 0$, $\psi v = 1$, and therefore $x = v$. If, without going so far, we put $\psi v = 1$, the general series (27) assumes the restricted form given above in (8).

Lest the reader suppose that the theorems here developed are not practically available, in comparison with those of Lagrange and Laplace, I add one or two examples. Let it be required to expand y^n in terms of x when $y = ze^{xy^k}$. Referring to (26), we have $\phi y = y^k$, so that the general (m^{th}) term of the expansion desired is

$$\mathrm{E}_0^n 0 \mathrm{E}_0^{km} z^0 0^{m-1} . x^m/m! = n\mathrm{E}_0^{n+km} z^0 0^{m-1} . x^m/m! = nz^{n+km}(n+km)^{m-1} . x^m/m!.$$

Again, to expand y^n when $y = z + xy^k \log y$, we shall have from (24), as the m^{th} term,

$$\mathrm{E}_0^n 0 \mathrm{D}_0^m \mathrm{E}_0^{km} z^{0-m}(0-1)(0-2) \ldots (0-m+1) . x^m/m!$$
$$= n\left(\frac{d}{d0}\right)^m z^{n+km-m+0}(0+n+km-1) \ldots (0+n+km-m+1)\, x^m/m!.$$

Apart from that in which $h = 1$, the chief symbolic special case of (18) is that in which $h = 0$. In this case $\mathrm{H} = (\mathrm{D} - u)/\phi\mathrm{E}$, where $\mathrm{D} = \log \mathrm{E} = \dfrac{d}{dy}$ (assuming the operations to be with respect to y), and $u = \log z$; also $x^{|m|} = x(\phi\mathrm{E}_x)^m e^{ux} x^{m-1}$. But in fact (18) is so sweepingly comprehensive that to dwell longer on special cases would only weary any reader not sufficiently interested to seek out the cases for himself.

It has been remarked that the coefficients of (24) and (27) must have the same value as those of Lagrange's theorem (23); that is, that

$$\left(\frac{d}{dz}\right)^{m-1}[(\phi z)^m f'z] = f\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^m z^{0-m}(0-1)(0-2) \ldots (0-m+1) \quad (28)$$
$$= f\mathrm{D}_0 0\,(\phi\mathrm{D}_0)^m e^{z0} 0^{m-1}. \quad (29)$$

These relations may, assuming f and ϕ developable by Maclaurin's theorem, be proved independently as follows. Since $f\text{E}_0 x^0 = fx$, and $\dfrac{d}{dz} fz = f\text{E}_0 z^{0-1}0$,

$$\begin{aligned}(\phi z)^m f'z &= (f\text{E}_0 z^{0-1}0) \, . \, (\phi\text{E}_0)^m z^0 \\ &= f\text{E}_0 0\, (\phi\text{E}_0)^m z^{0-1}; \qquad (30)\end{aligned}$$

for, if the general term in $f\text{E}$ is $a\text{E}^\alpha$ and that in $(\phi\text{E})^m$ is $b\text{E}^\beta$, the general term of the second member is $(a\text{E}^\alpha z^{0-1}0) \, . \, b\text{E}^\beta z^0 = ab\alpha\, z^{\alpha-1+\beta}$, and this is the general term of the final form. If we differentiate both sides of (30) $m-1$ times with respect to z, we obtain (28). Again, the general term of $f\text{D}_0 0\, (\phi\text{D}_0)^m e^{z0}$ is $a\text{D}_0^\alpha 0 b\text{D}_0^\beta e^{z0}$ $= \alpha a \text{D}_0^{\alpha-1} b\text{D}_0^\beta e^{z0} = \alpha ab e^{z0} z^{\alpha+\beta-1} = \alpha ab z^{\alpha+\beta-1}$, so that (29) may be proved similarly.